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A CONSEQUENCE OF THE INVARIANCE OF THE GAUSS PRINCIPLE*

V.A. VUJICIC

The invariant form of the Gauss principle of least compulsion in the space of positions of a system with constraints (some of which may be non-holonomic) is considered. A modified construction of the compulsion function in configuration space is proposed. The modified expression contains information on the constraints. From the complete system of differential equations, equations are obtained for finding the reactions of the constraints. An example of the use of this approach is given.

Manay authors, see /1, 2/, have considered the analytic form of the Gauss principle. However, there is still no standard treatment of the principle in analytical dynamics. For instance, it is said in /3/, p.192, that "the Gauss principle ... does not have the analytical advantages of other principles", and "is of less value than the principle at least action (/3/, p.134). Other authors (/4/, p.219) say that the "Gibbs-Appell equations (with which the Gauss principle is closely linked) represent the simplest and at the same time the most general form of the equations of motion". Yet, though these equations are closely linked with the principle of least compulsion, they do not contain the compulsion function

$$Z=\frac{1}{2}\sum_{\nu} m_{\nu}\left(x^{\cdot\cdot}-\frac{X_{\nu}}{m_{\nu}}\right)^{2} \tag{0.1}$$

but the Gibbs-Appell function

$$S=\frac{1}{2}\sum_{\nu} m_{\nu}x^{\cdot\cdot 2} \tag{0.2}$$

where m_ν is the mass of the ν-th point of the system, $x_\nu^{\cdot\cdot}$ are the coordinates of the acceleration vector, and X_ν are the coordinate of the vector of forces in a rectangular orthogonal system of coordinates.

Apart from these inconsistencies, there are difficulties in introducing the generalized Lagrange coordinates, in which

$$Z=S-Q_{\alpha}q^{\cdot\cdot\alpha} \tag{0.3}$$

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where Q_α are the generalized forces. Functions (0.1) and (0.3) are not the same and are not invariant.

The quadratic form of the Gauss compulsion in configuration space is obtained in /5/:

$$2Z^* = g_{\alpha\beta}(a^\alpha - Q^\alpha)(a^\beta - Q^\beta) \tag{0.4}$$

where $g_{\alpha\beta}$ are the coordinates of the metric tensor, a^α are the coordinates of the acceleration vector, and Q_α are the corresponding generalized forces. From the Gauss principle we easily obtian the differential equations of motion for a holonomic mechanical system:

$$\partial Z^*/\partial a^\alpha = 0 \tag{0.5}$$

or a non-holonomic mechanical system:

$$\partial Z^*/\partial a^\mu + c_\mu{}^\sigma \partial Z^*/\partial a^\sigma = 0 \tag{0.6}$$

Here,

$$a^\alpha = Dq^{\cdot\alpha}/dt = q^{\cdot\cdot\alpha} + \Gamma^\alpha_{\beta\gamma} q^{\cdot\beta} q^{\cdot\gamma} \tag{0.7}$$

But (0.4) is not the same as the Gauss compulsion (0.1), since (0.4) does not contain components of the accelerations $\mathbf{a}_\nu$ of vectors of the forces $\mathbf{F}_\nu$, orthogonal to the configuration space.

1. Consider the motion of a system of N practicles with masses m_ν ($\nu = 1, \ldots, N$), constrained by ideal scleronomic holonomic constraints

$$f_\sigma(\mathbf{r}_1, \ldots, \mathbf{r}_N) = 0 \qquad (\sigma = 1, \ldots, k) \tag{1.1}$$

where $\mathbf{r}_\nu$ is the position vector of the point M_ν. Let $x^1, x^2, \ldots x^{3N}$ be the particle coordinates in a curvilinear system of coordinates. Accordingly, the masses of particles M_ν are naturally written as $m_{3\nu-2} = m_{3\nu-1} = m_{3\nu}$, and the constraints (1.1) as

$$f_\sigma(x^1, x^2, \ldots, x^k, x^{k+1}, \ldots x^{3N}) = 0 \tag{1.2}$$

or in parametric form

$$\mathbf{r}_\nu = \mathbf{r}_\nu(x^1, \ldots x^{3N})\big|_{x^\sigma = x^\sigma(x^{k+1}, \ldots, x^{3N})}$$

We agree to denote k coordinates by $x^1, \ldots, x^k$, and the rest by q^α ($\alpha = 1, \ldots, n = 3N - k$), so that

$$\mathbf{r}_\nu = \mathbf{r}_\nu(q^1, \ldots, q^m; x^1, \ldots, x^k) \tag{1.3}$$

This means that the dimensionality of the configuration space E_{n+k} is $n+k$. Denote the basis vectors of the subspace $R_n \subset E_{n+k}$ by $\mathbf{g}_{\nu\alpha} = \partial\mathbf{r}_\nu/\partial q^\alpha$ and of the subspace $E_k \subset E_{3N}$ by $\mathbf{e}_{\nu i} = \partial\mathbf{r}_\nu/\partial x^i$ ($\alpha = 1, \ldots, n$; $i = 1, \ldots, k$).

The coordinate system is chosen so that the equations of the constraints are

$$f_\sigma = x_\sigma - c_\sigma = 0 \quad (c_\sigma = \text{const}) \tag{1.4}$$

In space E_{n+k} there act at point M_ν, in addition to the active forces $\mathbf{F}_\nu$, the imposed constraints (1.2); we replace then by the reactions $\mathbf{R}_\nu$. The principal vector of the forces at the point M_ν is then

$$\mathbf{F}_\nu{}^\circ = \mathbf{F}_\nu + \mathbf{R}_\nu \tag{1.5}$$

We resolve the acceleration and force vectors into their components in the coordinate system:

$$\mathbf{a}_\nu = a^\alpha \mathbf{g}_{\nu\alpha} + a_i \mathbf{e}_\nu{}^i \tag{1.6}$$

$$\mathbf{F}_\nu/m_\nu = Q^\alpha \mathbf{g}_{\nu\alpha} + F_i \mathbf{e}_\nu{}^i, \quad \mathbf{R}_\nu/m_\nu = R_i \mathbf{e}_\nu{}^i$$

where Q^α are the contravariant coordinates of the generalized forces in the n-dimensional configuration space M_n, F_i are the covariant coordinates of the forces in subspace E_k, and a^α, a_i are the corresponding coordinates of the acceleration vectors.

By definition, the Gauss compulsion is

$$Z = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \left(\mathbf{a}_\nu - \frac{\mathbf{F}_\nu{}^\circ}{m_\nu}\right)^2 \tag{1.7}$$

In the orthogonal Cartesian system of coordinates y^1, y^2, y^3 with basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, in which $\mathbf{F}_\nu{}^\circ = Y_\nu{}^s \mathbf{e}_s$, $\mathbf{a}_\nu = y_\nu{}^{\cdot\cdot s}\mathbf{e}_s$, compulsion (1.9) becomes

$$Z = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \sum_{s=1}^{3}\left(y_\nu^{\cdot\cdot s} - \frac{Y_\nu^{s}}{m_\nu}\right)^2$$

or in index notation

$$2Z = \delta_{\chi\varkappa}(y^{\cdot\cdot\chi} - Y^{\chi})(y^{\cdot\cdot\varkappa} - Y^{\varkappa}) \tag{1.8}$$

$$\delta_{\chi\varkappa} = m_\chi \mathbf{e}_\chi\cdot\mathbf{e}_\varkappa = \begin{cases} 0, & \chi\neq\varkappa \\ m_\chi, & \chi=\varkappa \end{cases}, \quad Y^{\chi} = \frac{Y_\nu^{\chi}}{m_\nu}$$

We substitute (1.6) into (1.7). Then,

$$2Z = \sum_{\nu=1}^{N} m_\nu\,[(a^\alpha - Q^\alpha)\,\mathbf{g}_{\nu\alpha} + (a_i - F_i - R_i)\mathbf{e}_\nu{}^i]^2 = \tag{1.9}$$

$$\sum_{\nu=1}^{N} m_\nu\,[(a^\alpha - Q^\alpha)\,\mathbf{g}_{\nu\alpha}]^2 + \sum_{\nu=1}^{N} m_\nu\,[(a_i - X_i)\mathbf{e}_\nu{}^i]^2$$

since the scalar product of the vectors $\mathbf{g}_{\nu\alpha}$ and $\mathbf{e}_\nu{}^i$ is zero and $X_i = F_i + R_i$.

We introduce the notation

$$g_{\alpha\beta} = \sum_\nu m_\nu \mathbf{g}_{\nu\alpha}\cdot\mathbf{g}_{\nu\beta}, \quad \mathbf{g}_{\nu\alpha} = \partial\mathbf{r}_\nu/\partial q^\alpha \tag{1.10}$$

$$g_{\alpha i} = \sum_\nu m_\nu \mathbf{g}_{\nu\alpha}\cdot\mathbf{e}_{\nu i}, \quad \mathbf{e}_{\nu i} = \partial\mathbf{r}_\nu/\partial x^i \tag{1.11}$$

$$\mathbf{e}^{ij} = \sum_\nu m_\nu \mathbf{e}_\nu{}^i\cdot\mathbf{e}_\nu{}^j \tag{1.12}$$

Since

$$\sum_\nu m_\nu\,[(a^\alpha - Q^\alpha)\mathbf{g}_{\nu\alpha}]^2 = \sum_\nu m_\nu \mathbf{g}_{\nu\alpha}\cdot\mathbf{g}_{\nu\beta}\,(a^\alpha - Q^\alpha)\,(a^\beta - Q^\beta)$$

$$\sum_\nu m_\nu\,[(a_i - X_i)\mathbf{e}_\nu{}^i]^2 = \sum_\nu m_\nu \mathbf{e}_\nu{}^i\cdot\mathbf{e}_\nu{}^j\,(a_i - X_i)\,(a_j - X_j)$$

the compulsion (1.9) can be reduced to the form

$$2Z = g_{\alpha\beta}\,(a^\alpha - Q^\alpha)\,(a^\beta - Q^\beta) + \mathbf{e}^{ij}\,(a_i - F_i - R_i)\,(a_j - F_j - R_j) \tag{1.13}$$

Expression (1.13) for the Gauss compulsion is equivalent to (1.7), (1.8); it is invariant, since (1.13) can be reduced to the form (1.8), i.e.,

$$2Z = g_{\chi\varkappa}\,(a^\chi - F^{\circ\chi})\,(a^\varkappa - F^{\circ\varkappa})$$

Notice that (1.13) includes all the coordinates of the vectors of the acceleration $a^\alpha = q^{\cdot\cdot\alpha} + \Gamma_{\beta\gamma}{}^\alpha q^{\cdot\beta} q^{\cdot\gamma}$ and forces Q^α, referred to the configuration space M_n, and the covariant coordinates of the acceleration $a_i = \mathbf{e}_{ij}\,(x^{\cdot\cdot j} + \Gamma_{\chi\varkappa}^{j}\, x^{\cdot\chi} x^{\cdot\varkappa})$ and force X_i, belonging to space E_k. Comparing (1.13) and (0.4), we can see that the invariant Gauss compulsion is more general than the compulsion (0.4) in the configuration space M_n, since $Z = Z^* + Z^{**}$, where

$$2Z^{**} = \mathbf{e}^{ij}\,(a_i - F_i - R_i)\,(a_j - F_j - R_j)$$

This addition to (0.4) does not change the differential equations of motion (0.5) and (0.6), since

$$\partial Z^*/\partial a^\alpha = \partial Z/\partial a^\alpha$$

Incidentally, the Gauss principle with compulsion (1.13) gives a greater number of equations than (0.6), i.e., greater than the number of degrees of freedom of the system. Starting from the fact that the first variation of the function Z in the Gauss sense ($\delta q = \delta q^{\cdot} = 0,\ \delta x = \delta x^{\cdot} = 0$) is zero:

$$\delta Z = (\partial Z/\partial a^\alpha)\,\delta a^\alpha + (\partial Z/\partial a_i)\,\delta a_i = 0 \tag{1.14}$$

we obtain n equations of the covariant type (0.5)

$$\partial Z/\partial a^\alpha = 0 \quad (\alpha = 1, \ldots, n) \tag{1.15}$$

and a further k equations of the contravariant type

$$\partial Z/\partial a_i = 0 \quad (i = 1, \ldots, k) \tag{1.16}$$

(see e.g., /6/, p.85).

Eqs.(1.16) are a consequence of relation (1.14), or due to the fact that the constraints are replaced by the forces $\mathbf{R}_\nu$ and all the variations of the accelerations δa^α and δa_i are assumed independent, or in accordance with the method of undetermined Lagrange multipliers in the light of constraints (1.2), satisfy the conditions

$$\sum_{\sigma=1}^{k} \lambda_\sigma \nabla_i f_\sigma \delta a^i = R_i \delta a^i = 0 \tag{1.17}$$

In this case the reactions R_i do not appear explicitly in (1.13), which takes the form

$$2Z_w = g_{\alpha\beta}(a^\alpha - Q^\alpha)(a^\beta - Q^\beta) + \mathbf{e}^{ij}(a_i - F_i)(a_j - F_j) \tag{1.18}$$

Using (1.14) and (1.17), we can write (1.16) as

$$\partial Z_w / \partial a_i = R_i \tag{1.19}$$

Since constraints of the type $x^i = c^i$ do not appear explicitly in (1.15), this system is equivalent to a Lagrange independent differential equation of the second kind. The remaining k equations of (1.16) and (1.19) form the part of the Lagrange equations of the first kind from which all the reactions of the holonomic constraints can be found.

2. The Gibbs-Appell function

$$S = \frac{1}{2} \sum_{\nu=1}^{N} m_\nu \frac{d\mathbf{v}_\nu}{dt} \cdot \frac{d\mathbf{v}_\nu}{dt} \tag{2.1}$$

can be reduced by means of (1.5), (1.9), (1.10), and (1.12), to the form

$$2S = g_{\alpha\beta} a^\alpha a^\beta + \mathbf{e}^{ij} a_i a_j \tag{2.2}$$

The Gauss compulsion (1.13) can be written as

$$2Z = g_{\alpha\beta} a^\alpha a^\beta + \mathbf{e}^{ij} a_i a_j - 2(g_{\alpha\beta} a^\alpha Q^\beta + \mathbf{e}^{ij} a_i X_j) + 2\Phi \tag{2.3}$$

$$2\Phi = g_{\alpha\beta} Q^\alpha Q^\beta + \mathbf{e}^{ij} X_i X_j = 2\Phi(t, q, \dot{q}; x)_{x=\mathrm{const}}$$

(2Φ is a quadratic form of the generalized forces, and is a function of the coordinates which generalize the velocities and time). Comparing (2.2) and (2.3), we obtain

$$Z = S - g_{\alpha\beta} a^\alpha Q^\beta - \mathbf{e}^{ij} a_i X_j + \Phi \tag{2.4}$$

From (1.15) we can now obtain the differential equations

$$\partial S / \partial a^\alpha - g_{\alpha\beta} Q^\beta = 0 \tag{2.5}$$

These are Appell's differential equations of motion

$$\partial S / \partial \ddot{q}^\alpha = Q_\alpha \tag{2.6}$$

for a holonomic system, since, by (0.7),

$$\frac{\partial S}{\partial \ddot{q}^\beta} = \frac{\partial S}{\partial a^\alpha} \frac{\partial a^\alpha}{\partial \ddot{q}^\beta} = \frac{\partial S}{\partial a^\alpha} \delta_\beta{}^\alpha = \frac{\partial S}{\partial a^\beta} \tag{2.7}$$

$$g_{\alpha\beta} Q^\beta = Q_\alpha$$

We cannot reduce (1.16) or (1.19) to Appell's form, since, for the constraints considered $\ddot{x}^i = 0$ and the partial derivative $\partial S / \partial \ddot{x}^i$ becomes meaningless.

3. In addition to the holonomic constraints (1.1) on the system of particles, let a further l non-holonomic constraints

$$\varphi_\mu(\mathbf{r}_1, \ldots, \mathbf{r}_N; \mathbf{v}_1, \ldots, \mathbf{v}_N) = 0 \quad (\mu = 1, \ldots, l < n) \tag{3.1}$$

be imposed, whence we obtain the relations for the accelerations

$$\mathbf{a}_\nu \cdot \mathrm{grad}_{\mathbf{v}_\nu} \varphi_\mu + \Theta(\mathbf{v}, \mathbf{r}) = 0$$

The equations in Gauss variations are

$$\delta \mathbf{a}_\nu \cdot \mathrm{grad}_{\mathbf{v}_\nu} \varphi_\mu = 0 \tag{3.2}$$

Substituting (1.6) into (3.2), we obtain the l relations

$$b_{\mu\alpha} \delta a^\alpha + b_\mu{}^i \delta a_i = 0 \tag{3.3}$$

$$b_{\mu\alpha} = \sum_{\nu=1}^{N} \mathbf{g}_{\nu\alpha} \cdot \mathrm{grad}_{\mathbf{v}_\nu} \varphi_\mu, \quad b_\mu{}^i = \sum_{\nu=1}^{N} \mathbf{e}_\nu{}^i \cdot \mathrm{grad}_{\mathbf{v}_\nu} \varphi_\mu$$

Using the method of undetermined Lagrange multipliers, we obtain from (3.3) and (1.14) the $n+k$ equations of motion of the non-holonomic system

$$\partial Z/\partial a^{\alpha} = \sum_{\mu=1}^{l} \lambda_\mu b_{\mu\alpha} \tag{3.4}$$

$$\partial Z/\partial a_i = \sum_{\mu=1}^{l} \lambda_\mu b_\mu{}^i \tag{3.5}$$

On rewriting (3.3) as

$$b_{\mu\rho}\delta a^\rho + b_{\mu p}\delta a^p + b_\mu{}^i \delta a_i \quad (\rho = 1, \ldots, l;\; p = l+1, \ldots, n)$$

we can find under the condition $|b_{\mu\rho}| \neq 0$ the dependent variations

$$\delta a^\mu = c_p{}^\mu \delta a^p + b^{\mu i}\delta a_i \tag{3.6}$$
$$c_p{}^\mu = b_{\rho p} b^{\rho\mu}, \quad b^{\mu i} = b_\nu{}^i b^{\rho\mu}; \quad b^{\rho\mu} = B_{\rho\mu}/|b_{\rho\mu}|$$

where $B_{\rho\mu}$ is the cofactor of element $b_{\rho\mu}$.

Substituting (3.6) into (1.14), we obtain

$$\left(\frac{\partial Z}{\partial a^p} + c_p{}^\mu \frac{\partial Z}{\partial a^\mu}\right)\delta a^p + \left(\frac{\partial Z}{\partial a_i} + b^{i\mu}\frac{\partial Z}{\partial a^\mu}\right)\delta a_i = 0$$

Hence follow the $n+k-l$ equations of motion of the non-homonomic system

$$\frac{\partial Z}{\partial a^p} + c_p{}^\mu \frac{\partial Z}{\partial a^\mu} = 0 \quad (\mu = 1, \ldots, l;\; p = l+1, \ldots, n) \tag{3.7}$$

$$\frac{\partial Z}{\partial a_i} + b^{i\mu}\frac{\partial Z}{\partial a^\mu} = 0 \quad (i = 1, 2, \ldots, k) \tag{3.8}$$

We can write Eqs.(3.7) and (3.8) with the aid of the Gibbs-Appell function

$$\frac{\partial S}{\partial a^p} = Q_p - c_p{}^\mu\left(\frac{\partial S}{\partial a^\mu} - Q_\mu\right) \tag{3.9}$$

$$\frac{\partial S}{\partial a_i} = F^i + R^i - b^{i\mu}\left(\frac{\partial S}{\partial a^\mu} - Q_\mu\right) \tag{3.10}$$

From these equations or the equations of motion (3.4) we obtain the explicit form of the Lagrange equations with multipliers

$$g_{\alpha\beta}(q^{\cdot\cdot\beta} + \Gamma^\beta_{\gamma\delta} q^{\cdot\gamma} q^{\cdot\delta}) = Q_\alpha + \sum_{\mu=1}^{l} \lambda_\mu b_{\mu\alpha}$$

Eqs.(3.5), (3.10) or (3.8) indicate the dependent of the reactions

$$R^i = \Gamma^i_{\alpha\beta} q^{\cdot\alpha} q^{\cdot\beta} - F^i + b^{i\mu}(a_\mu - Q_\mu)$$

on the multipliers $b^{i\mu}$ of the non-holonomic constraints (3.6). Clearly, if all the elements $b^{i\mu}$ are zero, then the non-holonomic constraints do not influence the size of the reactions of the holonomic constraints.

As an exmaple, consider the motion of a heavy sphere of mass m and radius r, with a fixed vertical cylinder of radius $R > r$ that cannot slip through onto its interior surface. In cylindrical coordinates $x^1 = \rho,\ q^2 = \chi,\ q^3 = \zeta$ with the Euler angles $q^4 = \varphi,\ q^5 = \psi,\ q^6 = \theta$, the non-holonomic constraints are given by

$$(R - r)\chi^{\cdot} + r\psi^{\cdot} + r\varphi^{\cdot}\cos\theta = 0$$
$$\zeta^{\cdot} - r\theta^{\cdot}\sin(\psi - \chi) - r\varphi^{\cdot}\sin\theta\cos(\psi - \chi) = 0$$

We define the generalized forces

$$F_\rho = 0;\ Q_\chi = Q_\varphi = Q_\psi = Q_\theta = 0;\ Q_\zeta = -mg$$

and the metric tensor

$$\|g_{\chi\kappa}\| = \begin{Vmatrix} A & 0 \\ 0 & B \end{Vmatrix}$$
$$A = \operatorname{diag}\{m, m\rho^2, m\}, \quad B = \begin{Vmatrix} J & J\cos\theta & 0 \\ J\cos\theta & J & 0 \\ 0 & 0 & J \end{Vmatrix}$$

where J is the moment of inertia of the sphere.

The compulsion function (1.13)

$$2Z = e^{11}(a_1 - F_1 - R_1)^2 + g_{\alpha\beta}(a^\alpha - Q^\alpha)(a^\beta - Q^\beta)$$

is equal to

$$(a_1 - R_1)^2/m + a_2 a^2 + (a_3 + mg)(a^3 + g) + a_4 a^4 + a_5 a^5 + a_6 a^6$$

The differential Eqs.(3.4) are obtained by simple differentiation; for example,

$$\partial Z/\partial a^2 = a_2 = \lambda_1 (R - r)$$

where $a_2 = \chi$. We obtain five covariant equations

$$a_\chi = \lambda_1 (R - r), \quad a_\zeta + mg = \lambda_2$$
$$a_\varphi = \lambda_1 r \cos\theta - \lambda_2 r \sin\theta \cos(\psi - \chi)$$
$$a_\psi = \lambda_1 r, \quad a_\theta = -\lambda_2 r \sin(\psi - \chi)$$

or, using Eqs.(3.7),

$$a_\chi = a_\psi (R - r)/r, \quad a_\theta = -(a_\zeta + mg) r \sin(\psi - \chi)$$
$$a_\varphi = a_\psi \cos\theta - (a_\zeta + mg) r \sin\theta \cos(\psi - \chi)$$

Here,

$$a_\chi = m(R - r)^2 \chi^{\cdot\cdot}, \quad a_\psi = J d/dt(\psi^\cdot + \varphi^\cdot \cos\theta), \quad a_\zeta = m\zeta^\cdot$$
$$a_\varphi = J d/dt(\varphi^\cdot + \psi^\cdot \cos\theta), \quad a_\theta = J(\theta^{\cdot\cdot} + \varphi^\cdot \psi^\cdot \sin\theta)$$

Eq.(3.5), corresponding to the coordinate $x^1 = \mathrm{const}$

$$\partial Z/\partial a_1 = e^{11}(a_1 - R_1) = \lambda_1 b_1{}^1 + \lambda_2 b_2{}^1$$

gives the expression for the reaction

$$R_1 = a_1 = a_\rho = -mR\chi^{\cdot 2}$$

Here we note that

$$b_1{}^1 = b_2{}^1 = 0, \; a_\rho = a_1 = \Gamma_{\alpha\beta,1} q^{\cdot\alpha} q^{\cdot\beta} = \Gamma_{22,1} q^{\cdot 2} q^{\cdot 2} = -m\rho\chi^\cdot\chi^\cdot$$

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